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Monotonicity Properties of the Zeros of Ultraspherical Polynomials*

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Let $x_{n,k}^{(\lambda)}$, $k=1, 2, ..., [n/2]$, denote the kth positive zero in increasing order of the ultraspherical polynomial $P_n^{(\lambda)}(x)$. We prove that the function $[\lambda+(2n^2+1)/(4n+2)]^{1/2}\, x_{n,k}^{(\lambda)}$ increases as λ increases for $\lambda>-1/2$. The proof is based on two integrals involved with the square of the ultraspherical polynomial $P_n^{(\lambda)}(x)$. © 1999 Academic Press

1. INTRODUCTION AND THE MAIN RESULTS

Let $x_{n,k}^{(\lambda)}$, $k=1, 2, ..., [n/2]$, denote the kth positive zero in increasing order of the ultraspherical polynomial $P_n^{(\lambda)}(x)$, $n=0, 1, 2, ...,$ $\lambda>-1/2$. A known result, due to Stieltjes [16; 17, Theorem 6.2.11.1], says that for any fixed $n\geqslant 2$ and k, $1\leqslant k\leqslant [n/2]$, the positive zeros $x_{n,k}^{(\lambda)}$ decrease as λ increases. In [14], A. Laforgia proved that the function $\lambda x_{n,k}^{(\lambda)}$ increases as λ increases at least for $0<\lambda<1$. In [1], S. Ahmed *et al.* have found the more general result, namely the function $[\lambda+(2n^2+1)/(4n+2)]^{1/2}\, x_{n,k}^{(\lambda)}$ increases as λ increases for $-1/2<\lambda\leqslant 3/2$. Then in [13], M. E. H. Ismail and J. Letessier formulated a conjecture in the form that $\sqrt{\lambda}\, x_{n,k}^{(\lambda)}$ increases as λ increases for $\lambda>0$. Later in [12] this was reformulated as the Ismail–Letessier–Askey conjecture (ILAC):

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ILAC *Conjecture.* Let $n > 2$ and $1 \leqslant k \leqslant [n/2]$, then the function $(\lambda+1)^{1/2}\, x_{n,k}^{(\lambda)}$ increases as λ increases for $\lambda > -1/2$.

This conjecture is supported by the following known facts:

(i) When $n = 2$, $x_{2,1}^{(\lambda)} = 1/\sqrt{2(\lambda+1)}$, and $n = 3$, $x_{3,1}^{(\lambda)} = \sqrt{3/2(\lambda+2)}$, from where the ILAC follows.

(ii) The above mentioned Ahmed–Muldoon–Spigler result implies the ILAC for $-1/2 < \lambda < 3/2$ and $n > 3$.

(iii) In [11], E. Ifantis and the second named author proved the ILAC for the largest positive zero $x_{n,[n/2]}^{(\lambda)}$ using a functional analytic technique.

(iv) Recently D. Dimitrov [2] proved the ILAC for all positive zeros $x_{n,k}^{(\lambda)}$ for $\lambda \in (-1/2, 9/2]$ and also for $\lambda \in (-1/2, 3/2+\nu)$ and $n > 1 + (\nu^2 + 3\nu + 3/2)^{1/2}$ where $\nu \in \mathbb{N}$. Moreover he proved this conjecture for the largest zero $x_{n,[n/2]}^{(\lambda)}$ as E. Ifantis and P. D. Siafarikas, using different method. Finally, D. Dimitrov announced in a review paper [3] that he proved the ILAC for the smallest positive zero $x_{n,1}^{(\lambda)}$ of $P_n^{(\lambda)}(x)$ for $\lambda \geqslant 2$.

Our contribution in this direction is the following.

THEOREM. *Let $n \geqslant 3$ and $1 \leqslant k \leqslant [n/2]$. Then the function $[\lambda + (2n^2/+1)/(4n+2)]^{1/2}\, x_{n,k}^{(\lambda)}$ increases as λ increases for $\lambda > -1/2$.*

Due to the fact that $(\lambda+a)/(\lambda+b)$ increases as λ increases provided $a < b$ and $\lambda + b > 0$, our Theorem implies the ILAC because $(2n^2+1)/(4n+2) > 1$ for $n \geqslant 3$.

For the proof of our Theorem we shall need the following definite integrals. Let $I_\nu = I_\nu(n, \lambda)$ be defined by

$$I_\nu = I_\nu(n, \lambda) = \int_{-1}^{1} (1-x^2)^{\lambda-\nu-1/2}\, [P_n^{(\lambda)}(x)]^2\, dx,$$

$$\lambda > \nu - \tfrac{1}{2}, \qquad \nu = 0, 1, 2. \tag{1.1}$$

Particularly, I_0 is well known [17, p. 80],

$$I_0(n, \lambda) = \frac{\pi\, 2^{1-2\lambda}\, \Gamma(n+2\lambda)}{n!\, (n+\lambda)[\Gamma(\lambda)]^2}, \qquad \lambda > -\frac{1}{2}, \qquad n = 0, 1, \ldots \tag{1.2}$$

because it plays a role in the theory of ultraspherical polynomials.

The formula for I_1 is less familiar,

$$I_1(n, \lambda) = \frac{\pi 2^{1-2\lambda} \Gamma(n+2\lambda)}{n!\,(\lambda - 1/2)[\Gamma(\lambda)]^2}, \qquad \lambda > \frac{1}{2}, \qquad n = 0, 1, ..., \tag{1.3}$$

but it is equivalent to [8, 281(9)] or [9, 7.314(1)] or to (3.9) in [7].

Finally, the case $v = 2$ is also not incorporated into the standard books on definite integrals:

$$I_2(n, \lambda) = \frac{\pi 2^{-2\lambda} \Gamma(n+2\lambda)}{n!\,[\Gamma(\lambda)]^2} \frac{(n+\lambda)^2 + \lambda^2 - \lambda - 1}{(\lambda + 1/2)(\lambda - 1/2)(\lambda - 3/2)},$$

$$\lambda > \frac{3}{2}, \qquad n = 0, 1, ... \tag{1.4}$$

In Section 3 we shall give a simple proof of the last two formulas.

Let us recall an asymptotic result on the zeros of the ultraspherical polynomials from [6],

$$x_{n,k}^{(\lambda)} = h_{n,k} \lambda^{-1/2} \left[1 - \frac{2n - 1 + 2h_{n,k}^2}{8\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \right] \qquad (\lambda \to \infty), \tag{1.5}$$

where $h_{n,k}$ denotes the corresponding zero of the Hermite polynomial $H_n(x)$. Then clearly,

$$\lim_{\lambda \to \infty} \left[\lambda + \frac{2n^2 + 1}{4n + 2} \right]^{1/2} x_{n,k}^{(\lambda)} = h_{n,k}$$

and combining this limit with the monotonicity stated in our Theorem, we conclude

COROLLARY. *For the positive zeros $x_{n,k}^{(\lambda)}$ of the ultraspherical polynomial $P_n^{(\lambda)}(x)$ the inequality*

$$\left[\lambda + \frac{2n^2 + 1}{4n + 2} \right]^{1/2} x_{n,k}^{(\lambda)} < h_{n,k} \qquad \textit{for} \quad \lambda > -\frac{1}{2}, \qquad k = 1, 2, ..., \left[\frac{n}{2}\right]$$

holds.

The zeros $h_{n,k}$ of the Hermite polynomial $H_n(x)$ have different asymptotic behavior near $x = 0$ or for large values of x when n tends to infinity [17, p. 130, 132],

$$\lim_{n\to\infty} \sqrt{2n+1}\, h_{n,1} = \begin{cases} \pi/2 & \text{if } n \text{ even,} \\ \pi & \text{if } n \text{ odd,} \end{cases} \tag{1.6}$$

$$h_{n,[n/2]} = \sqrt{2n+1} - \mathcal{O}(n^{-1/6}). \tag{1.7}$$

Applying this information, we get for the zero $x_{n,1}^{(\lambda)}$:

$$\sqrt{\lambda + \frac{2n^2+1}{4n+2}}\, x_{n,1}^{(\lambda)} = h_{n,1}\left[1 - \frac{4(2n+1)\, h_{n,1}^2 - 3}{8(2n+1)\,\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right]$$

$$\text{for} \quad \lambda \gg n \gg 1$$

hence by (1.6) the result formulated in our Theorem is rather sharp, in other words, it would be a hard task to improve this Theorem keeping it valid for *all* n, for *all* $\lambda > -\frac{1}{2}$, and for *all* positive zeros of $P_n^{(\lambda)}(x)$.

For the largest zero of the ultraspherical polynomials our result is not so sharp. In that case the asymptotic formula for $x_{n,k}^{(\lambda)}$ and (1.7) give

$$x_{n,[n/2]}^{(\lambda)} = \{\sqrt{2n+1} - \mathcal{O}(n^{-1/6})\}\, \frac{1}{\sqrt{\lambda + (3n + \mathcal{O}(n^{1/3}))/2 + \mathcal{O}(1/\lambda)}}$$

$$\sim \frac{\sqrt{2n}}{\sqrt{\lambda + (3/2)\, n}}, \qquad \lambda \gg n \gg 1.$$

Now this relation can be compared favorable with another inequality from [5]:

$$x_{n,[n/2]}^{(\lambda)} < \frac{\sqrt{n^2 + 2n\lambda}}{n+\lambda}$$

because

$$\frac{\sqrt{2n}}{\sqrt{\lambda + (3/2)\, n}} - \frac{\sqrt{n^2+2n\lambda}}{n+\lambda} = \mathcal{O}\left(\left(\frac{n}{\lambda}\right)^{5/2}\right) \qquad \text{for} \quad \lambda \gg n \gg 1.$$

Another corollary would be formulated by using our Theorem if we compare a positive zero $x_{n,k}^{(\lambda)}$ with $x_{n,k}^{(\lambda_0)}$ where λ_0 is chosen particularly; e.g., for $\lambda_0 = 0$, $C_n^{(0)}(x) = (2/n)\, T_n(x)$, hence the zeros are $\cos((2m-1)/2n)\,\pi$, for $\lambda_0 = 1$: $C_n^{(1)}(x) = \text{const}\; U_n(x)$ with zeros $\cos(m/n+1)\,\pi$. Then

$$\left(\lambda + \frac{2n^2+1}{4n+2}\right)^{1/2} x_{n,k}^{(\lambda)} \lessgtr \left(\lambda_0 + \frac{2n^2+1}{4n+2}\right)^{1/2} x_{n,k}^{(\lambda_0)} \qquad \text{if} \quad \lambda \lessgtr \lambda_0.$$

2. PRELIMINARIES

Our proof is based on the application of the generalized Richardson formula established in [7]: if $U(t) = U(\lambda, t)$ is a solution of

$$\frac{d^2U(t)}{dt^2} + R(\lambda, t)\, U(t) = 0$$

with the condition either $U(0) = 0$ or $(d/dt)\, U(0) = 0$, and $c(\lambda)$ is a zero of $U(\lambda, t) = 0$, then $c(\lambda)$ is a differentiable function and

$$\left[\frac{d}{dt}\, U(\lambda, c(\lambda))\right]^2 \frac{dc(\lambda)}{d\lambda} = -\int_0^{c(\lambda)} \frac{\partial R(\lambda, t)}{\partial \lambda}\, U^2(\lambda, t)\, dt \tag{2.1}$$

provided $(\partial/\partial\lambda)\, R(\lambda, t)$ is continuous on $[0, c(\lambda)]$. This formula was found by Richardson [15] in the case $U(0) = 0$.

It is well known that the ultraspherical polynomial $P_n(x) = P_n^{(\lambda)}(x)$ of degree n satisfies the second order differential equation [17, p. 80]

$$(1 - x^2)\, y'' - (2\lambda + 1)\, xy' + n(n + 2\lambda)\, y = 0 \tag{2.2}$$

and the polynomials $\{P_n^{(\lambda)}(x)\}_{n=0}^{\infty}$ are orthogonal on the interval $[-1, 1]$ with the weight function $(1 - x^2)^{\lambda - (1/2)}$ which means that

$$\int_{-1}^{1} (1 - x^2)^{\lambda - (1/2)}\, P_n^{(\lambda)}(x)\, P_m^{(\lambda)}(x)\, dx = 0 \qquad \text{if} \quad n \neq m.$$

On the other hand, by the symmetry relation $P_n(-x) = (-1)^n\, P_n(x)$ we have $P_n(0)\, P'_n(0) = 0$. The function $u(x) = (1 - x^2)^{\lambda/2 + 1/4}\, P_n^{(\lambda)}(x)$ satisfies the Sturm-Liouville differential equation

$$\frac{d^2u(x)}{dx^2} + Q(\lambda, x)\, u(x) = 0,$$

where

$$Q(\lambda, x) = \frac{(n + \lambda)^2}{1 - x^2} + \frac{1/2 + \lambda - \lambda^2 + x^2/4}{(1 - x^2)^2}.$$

By substitution $t = f(\lambda)x$ with $f(\lambda) = [\lambda + (2n^2 + 1)/(4n + 2)]^{1/2}$, the function $U(t) = u(x)$ satisfies the differential equation

$$\frac{d^2U(t)}{dt^2} + R(\lambda, t)\, U(t) = 0$$

with

$$R(\lambda, t) = [f(\lambda)]^{-2}\, Q(\lambda, t/f(\lambda)).$$

Making the substitutions $t^2=\tau$ and $f^2=\varphi(\lambda)=\lambda+(2n^2+1)/(4n+2)$, we have

$$S(\lambda, \varphi(\lambda), \tau)=R(\lambda, t)=\frac{(n+\lambda)^2}{\varphi(\lambda)-\tau}+\frac{\varphi(\lambda)(1/2+\lambda-\lambda^2)+\tau/4}{(\varphi(\lambda)-\tau)^2}.$$

It is clear that $U(0)\,U'(0)=0$ so we can apply the Richardson formula (2.1). Thus we have to calculate the derivative $(d/d\lambda)\,S(\lambda, \varphi(\lambda), \tau)$:

$$(\varphi(\lambda)-\tau)^3\,\frac{d}{d\lambda}\,S(\lambda, \varphi(\lambda), \tau)=A\tau^2+B\tau+C, \tag{2.3}$$

where

$$\begin{aligned} A&=2(n+\lambda),\\ B&=-(4n+2\lambda+1)\,\varphi(\lambda)+\lambda^2-\lambda-1+(n+\lambda)^2,\\ C&=(2n+1)\,\varphi^2(\lambda)+[\lambda^2-\lambda-1/2-(n+\lambda)^2]\,\varphi(\lambda). \end{aligned}$$

By definition of $\varphi(\lambda)$ we have $C=0$. Let $\tau_0=-B/A$. Then $\tau_0\geqslant\varphi(\lambda)$ if $\lambda\in(-\frac{1}{2}, \frac{3}{2}]$, and $0<\tau_0<\varphi(\lambda)$ if $\lambda>\frac{3}{2}$, hence

$$\frac{dS(\lambda, \varphi(\lambda), \tau}{d\lambda}\begin{cases} <0 & \text{for}\quad 0<\tau<\varphi(\lambda) & \text{if}\quad \lambda\in\left(-\frac{1}{2}, \frac{3}{2}\right],\\ <0 & \text{for}\quad 0<\tau<\tau_0 & \text{if}\quad \lambda>\frac{3}{2},\\ >0 & \text{for}\quad \tau_0<\tau<\varphi(\lambda) & \text{if}\quad \lambda>\frac{3}{2}. \end{cases} \tag{2.4}$$

Now let $c(\lambda)=f(\lambda)\,x_{n,k}^{(\lambda)}$ be a positive zero of $U(\lambda, t)=0$, hence by (2.1) the sign of $dc(\lambda)/d\lambda$ is determined by the integral

$$\Phi(c)=-\int_0^c\frac{dS(\lambda, \varphi(\lambda), t^2)}{d\lambda}\,U^2(\lambda, t)\,dt, \qquad 0<c<f(\lambda). \tag{2.5}$$

According to (2.4), this integral is always positive if $\lambda\in(-\frac{1}{2}, \frac{3}{2}]$ because the integrand itself is negative. This fact was already exploited in [4]. Our main observation here is that this integral is positive also for $\lambda>3/2$. By (2.4) the function $\Phi(c)$ increases as c increases for $0<c<\sqrt{\tau_0}$, and $\Phi(c)$ attains its maximum at $c=\sqrt{\tau_0}$, then $\Phi(c)$ is decreasing on $(\sqrt{\tau_0}, f(\lambda))$. At the endpoint $c=f(\lambda)$ it is vanishing, i.e., we have the following result.

LEMMA. *The function $\Phi(c)$ defined in (2.5) is positive for $0<c<f(\lambda)$ and $\Phi(0)=\Phi(f(\lambda))=0$.*

It is clear that Lemma implies our Theorem. Therefore by (2.4), (2.5) we have "only" to show that

$$\int_0^{f(\lambda)} \frac{dS(\lambda, \varphi(\lambda), t^2)}{d\lambda} U^2(\lambda, t)\, dt$$
$$= f^{-3}(\lambda) \int_0^1 \frac{A\varphi(\lambda)\, x^4 + Bx^2}{(1-x^2)^3} (1-x^2)^{\lambda+1/2} P_n^2(x)\, dx = 0 \qquad \text{for} \quad \lambda > \frac{3}{2}. \tag{2.6}$$

3. PROOFS

Proof of Relations (1.3), (1.4).[1]

For $v = 1, 2$, $\lambda > v - \frac{1}{2}$ we have by (1.1)

$$\begin{aligned}
(2\lambda - 2v + 1)(I_v - I_{v-1})
&= (2\lambda - 2v + 1) \int_{-1}^1 (1-x^2)^{\lambda - v - 1/2}\, x^2 P_n^2(x)\, dx \\
&= -\int_{-1}^1 [(1-x^2)^{\lambda - v + 1/2}]'\, x P_n^2(x)\, dx \\
&= [-(1-x^2)^{\lambda - v + 1/2}\, x P_n^2(x)]_{-1}^1 \\
&\quad + \int_{-1}^1 (1-x^2)^{\lambda - v + 1/2} [P_n^2(x) + 2x P_n(x)\, P_n'(x)]\, dx \\
&= I_{v-1} + 2 \int_{-1}^1 (1-x^2)^{\lambda - v + 1/2}\, x P_n'(x)\, P_n(x)\, dx.
\end{aligned} \tag{3.1}$$

In case $v = 1$ the last integral can be easily determined: since $xP_n'(x) = nP_n(x) + \sum_{i=0}^{n-2} c_{n,i} x^i$ where $c_{n,i}$ are constant, the orthogonality of the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ gives for this integral

$$\begin{aligned}
\int_{-1}^1 (1-x^2)^{\lambda - 1/2}\, x P_n'(x)\, P_n(x)\, dx
&= \int_{-1}^1 (1-x^2)^{\lambda - 1/2} \left(nP_n(x) + \sum_{i=0}^{n-2} c_{n,i} x^i \right) P_n(x)\, dx \\
&= n \int_{-1}^1 (1-x^2)^{\lambda - 1/2} P_n^2(x)\, dx = nI_0,
\end{aligned}$$

[1] As a referee has pointed out, these relations—even more general—can be deduced using formulas (16.3.16) or (16.3.17) of [8].

hence by (3.1)

$$(2\lambda-1)\, I_1 = 2(n+\lambda)\, I_0 \qquad \text{for} \quad \lambda > \tfrac{1}{2}. \tag{3.2}$$

By (1.2), (3.2) the formula (1.3) follows.

In case $v=2$ we proceed in the following way. By (2.2) we have for $y = P_n(x)$

$$(1-x^2)\, P_n''(x) - (2\lambda+1)\, xP_n'(x) + n(n+2\lambda)\, P_n(x) = 0.$$

Multiply this identity by $(1-x^2)^{\lambda-3/2}\, P_n(x)$ and integrate it over $[-1, 1]$ we obtain

$$(2\lambda+1) \int_{-1}^{1} (1-x^2)^{\lambda-3/2}\, xP_n'(x)\, P_n(x)\, dx$$

$$= -\int_{-1}^{1} (1-x^2)^{\lambda-1/2}\, P_n''(x)\, P_n(x)\, dx + n(n+2\lambda)\, I_1,$$

where the integral on the right hand side is zero because of the orthogonality. Hence we get by (3.1)

$$2(\lambda-\tfrac{3}{2})(\lambda+\tfrac{1}{2})\, I_2 = [(n+\lambda)^2 + \lambda^2 - \lambda - 1]\, I_1 \tag{3.3}$$

and (1.4) follows from (1.3). ∎

Proof of Relation (2.6). Taking into account of the actual values $A = 2(n+\lambda)$ and B in (2.3), we have

$$A\varphi(\lambda)\, x^4 + Bx^2 = \varphi(\lambda)[2(n+\lambda)(1-x^2)^2 - (2\lambda-1)(1(-x^2)]$$

$$+\left[2\left(\lambda+\frac{1}{2}\right)\left(\lambda-\frac{3}{2}\right) - [(n+\lambda)^2+\lambda^2-\lambda-1](1-x^2)\right].$$

Then by (1.1) we obtain for the integral in (2.6)

$$\frac{1}{2}\int_{-1}^{1} \frac{A\varphi(\lambda)\, x^4 + Bx^2}{(1-x^2)^3}\, (1-x^2)^{\lambda+1/2}\, P_n^2(x)\, dx$$

$$= \varphi(\lambda)[2(n+\lambda)\, I_0 - (2\lambda-1)\, I_1]$$

$$+\left[2\left(\lambda+\frac{1}{2}\right)\left(\lambda-\frac{3}{2}\right) I_2[(n+\lambda)^2+\lambda^2-\lambda-1]\, I_1\right],$$

hence the relations (3.2), (3.3) imply that this integral is zero. ∎

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Note added in proof. In 1984 A. Laforgia [18] conjectured that the function $\lambda x_{n,k}^{(\lambda)}$ increases for $\lambda > 0$ and this conjecture initiated the ILAC.

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